

Predictable nonwandering localization of covariant Lyapunov vectors for scale-free networks of chaotic maps

Pavel V. Kuptsov^{1,*} and Anna V. Kuptsova¹

¹*Institute of electronics and mechanical engineering,*

Yuri Gagarin State Technical University of Saratov, Politekhnikeskaya 77, Saratov 410054, Russia

(Dated: February 21, 2014)

For scale-free networks of Hénon maps we show that the first covariant Lyapunov vectors demonstrate high nonwandering localization. The nodes of localization are not synchronized with others, and the distributions of square deviations of dynamical variables from their neighborhood have identical power law shapes for all of such nodes. The revealed features of the localization nodes allow to find them without computing of the vectors.

PACS numbers: 89.75.Hc, 05.45.Ra, 05.45.Xt, 05.45.Jn, 05.45.Pq, 05.45.-a

Introduction. — Localization properties of Lyapunov vectors in spatio-temporal chaotic systems attract a permanent interest since the early works [1–3], and recently it was renewed due to the discovery of algorithms for covariant Lyapunov vectors (CLVs) [4, 5]. The evolution of the vectors is governed by linear equations under chaotic forcing, so that their localization can be treated as a sort of Anderson localization [2]. The localization sites indicate unstable areas of a system, that, in particular, important for atmosphere dynamics prediction [6]. For homogeneous systems the localization sites of the covariant vectors wander irregularly so that their dynamics can be described by stochastic equation of Kardar-Parisi-Zhang [7, 8]. In contrast, the localization positions in inhomogeneous systems are pinned at certain fixed positions [9].

In this Letter we apply the apparatus of CLVs to the analysis of scale-free networks of chaotic maps. We show that the first CLVs for the these generalized extended systems are highly localized, and the localization is nonwandering, i.e., nonzero sites of the vectors always belong to a permanent subset of network nodes set. Moreover these nodes have specific topological and dynamical properties so that they can be identified without computing the CLVs. This is a remarkable example of explicit relations between the system topology, its phase space dynamics, and the associated tangent space dynamics of CLVs.

Model system. — We consider a network of Hénon maps build as a generalization of the Hénon chain from Ref. [10]:

$$\begin{aligned} x_n(t+1) &= \alpha - [x_n(t) + \epsilon g_n(t)]^2 + y_n(t), \\ y_n(t+1) &= \beta x_n(t), \end{aligned} \quad (1)$$

$$g_n(t) = \sum_{j=1}^N \frac{a_{nj}}{k_n} x_j(t) - x_n(t), \quad k_n = \sum_{j=1}^N a_{jn}, \quad (2)$$

where N is the number of network nodes, $t = 0, 1, 2, \dots$ is discrete time, $a_{nj} \in \{0, 1\}$, $a_{nn} = 0$ are the elements

of the $N \times N$ adjacency matrix \mathbf{A} , and k_n is degree of the n th node, i.e., the number of its connections. $\alpha = 1.4$ and $\beta = 0.3$ are the parameters, controlling local dynamics, and $\epsilon \in [0, 1]$ is the coupling strength. The system (1) is time-reversible: $x_n(t) = y_n(t+1)/\beta$, $y_n(t) = -\alpha + [y_n(t+1) + \epsilon h_n(t+1)]^2/\beta^2 + x_n(t+1)$, where $h_n(t) = \sum_{j=1}^N \frac{a_{nj}}{k_n} y_j(t) - y_n(t)$.

In what follows we will consider the instantaneous square deviations of nodes from their neighborhood taking into account both forward and backward time dynamics:

$$\tau_n(t) = g_n^2(t) + h_n^2(t). \quad (3)$$

We consider scale-free networks with N nodes and $N - 1$ connections generated via a stochastic process described in Ref. [11]. The node degree distribution of such networks has a power law shape $P(k) \sim k^{-3}$.

During the computations some CLVs sites can intermittently be dropped to machine zeros. Given identical initial conditions and two different algorithmic implementations of Eq. (1) one can observe this drops only for one of them, while the other one evolves flawlessly. This numerical artifact results in physically meaningless peaks in the Lyapunov exponent spectrum. To cure it we added a random variable of the amplitude 10^{-10} to x_n and y_n at each tenth iteration. Numerical problems with inhomogeneous chaotic system are also discussed in Ref. [9].

The dynamics of the system (1) is generally chaotic. Though the full synchronization of all nodes is not observed, the nodes can form clusters of synchronized oscillations. Both full and phase synchronization is possible. The former stands for the identity of the synchronized variables, and the later implies the coincidence of positions of minima and maxima of synchronized time series. Below we will consider only the phase synchronization.

To detect the clusters we find phase distances between nodes as suggested in Ref. [12]. Given a time interval \mathcal{T} , count the numbers ν_m and ν_n of local minima of x_m and x_n , respectively, and also find the number ν_{mn} of simultaneous minima of x_m and x_n . Then the phase distance is computed as $d_{mn} = 1 - \nu_{mn}/\max(\nu_m, \nu_n)$. It vanishes

* Corresponding author. Electronic address: p.kuptsov@rambler.ru

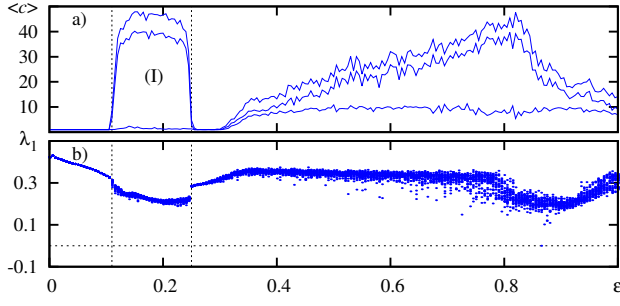


Figure 1. (color online) a) Average sizes $\langle c \rangle$ of three largest phase synchronization clusters. b) The first Lyapunov exponent. For both panels 25 computations with different \mathbf{A} are performed at each ϵ . $\mathcal{T} = 1000$, $N = 128$. Lines at $\epsilon = 0.11$ and 0.25 delimit the area of interest.

when all the minima of x_m and x_n occur simultaneously, and $d_{mn} = 1$ when none of them match. To identify the clusters we find the matrix of d_{mn} and build a graph whose n th and m th nodes are connected if $d_{mn} = 0$. Finding connected components of this graph we identify the clusters of phase synchronized nodes.

Figure 1(a) shows the sizes of synchronization clusters against ϵ , and Fig. 1(b) represents the first Lyapunov exponent. The synchronization occurs in the area (I) at $0.11 < \epsilon < 0.25$. Notice that the corresponding values of λ_1 are smaller here than in the surrounding areas. There are two large clusters each including approximately $30\% \div 40\%$ of network nodes. The rest of the nodes either oscillate separately or are connected to very small clusters of 2-3 elements. To the left of the area (I) any clustering is absent, while to right the clusters can be sufficiently large. In this Letter we will restrict ourselves with the area (I).

CLVs and localization measures. — We compute CLVs using the numerical method suggested in Ref. [5] with the improvements from Refs [8, 13]. Notice that there exists another method [4]. Given elements of the i th CLV γ_{ji} , where $j = 1, \dots, 2N$ enumerates the vector sites, we compute CLV element related to the n th node as $p_{ni} = \gamma_{2n-1,i}^2 + \gamma_{2n,i}^2$, where $\sum_{n=1}^N p_{ni} = 1$ for any i .

A vector with unit sum of elements is said to be localized if almost all its elements are zeros except one or a few ones, whereas a delocalized vector have nonzero and approximately equal elements. Two measures are usually employed in literature to quantify the Lyapunov vector localization. The inverse participation ratio $Y_2^{(i)} = \sum_{n=1}^N p_{ni}^2$ is $O(1)$ for localized vectors and $O(1/N)$ for delocalized ones [1, 14]. The exponential normalized entropy $W^{(i)} = \exp\left(-\sum_{n=1}^N p_{ni} \ln p_{ni}\right)/N$ is $O(1)$ for delocalized vectors and $O(1/N)$ for localized ones [3, 15]. One can easily see, however, that these two measures are related to the Rényi entropy [16] $H_\alpha^{(i)} = \ln\left(\sum_{n=1}^N p_{ni}^\alpha\right)/(1-\alpha)$, namely, $H_2^{(i)} = -\ln Y_2^{(i)}$ and $H_1^{(i)} = \ln(NW^{(i)})$.

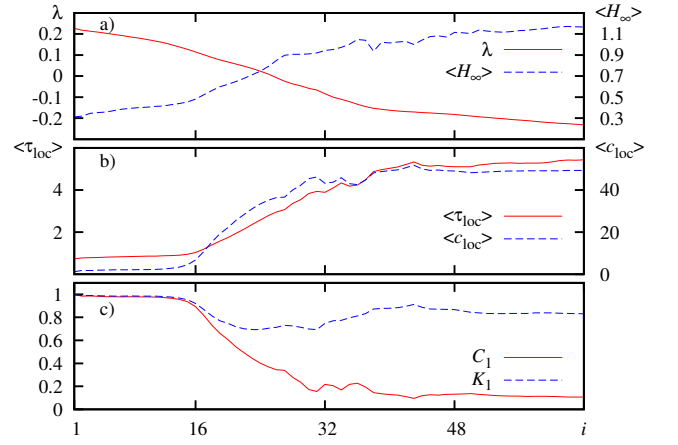


Figure 2. (color online) Lyapunov exponents λ , average entropy $\langle H_\infty \rangle$ and related values $\langle \tau_{\text{loc}} \rangle$, $\langle c_{\text{loc}} \rangle$, C_1 , and K_1 computed at localization nodes as explained in text. Total averaging time is 10^5 , where 10^2 cluster detections are performed for trajectory cuts of $\mathcal{T} = 10^3$ steps. $N = 128$, $\epsilon = 0.17$.

This is a smooth function that decays from $\ln N$ at $\alpha = 0$ to $H_\infty^{(i)} = -\ln p_{n_{\text{loc}}^{(i)}}$ at $\alpha \rightarrow \infty$, where $p_{n_{\text{loc}}^{(i)}} = \max\{p_{ni} | 1 \leq n \leq N\}$. $H_\infty^{(i)} = 0$ for fully localized vectors and $H_\infty^{(i)} = \ln N$ for fully delocalized ones. Since tests indicate that $Y_2^{(i)}$, $W^{(i)}$ and $H_\infty^{(i)}$ behave qualitatively similar, we will use $H_\infty^{(i)}$ as a CLV localization measure. It provides two profits: it takes less computational time, and the localization node $n_{\text{loc}}^{(i)}$ is determined in a natural way.

Localization properties. — To characterize localization properties of CLVs we move along a trajectory and for each considered vector find the entropy $H_\infty^{(i)}$, the corresponding localization nodes $n_{\text{loc}}^{(i)}$, and the deviations τ_n and cluster sizes c_n at $n = n_{\text{loc}}^{(i)}$. Being averaged in time these values are shown in Fig. 2(a,b) together with the corresponding Lyapunov exponents. Panel (c) represents the frequency C_1 of vector localization at separated nodes, i.e., at nodes that are not synchronized with any cluster and thus have $c_n = 1$, and the relative frequency K_1 of localization at nodes with $k_n = 1$.

Small entropy $\langle H_\infty^{(i)} \rangle$ of the first vectors in Fig. 2(a) indicates their high localization, while plots for $\langle \tau_{\text{loc}} \rangle$ and $\langle c_{\text{loc}} \rangle$ in Fig. 2(b) reveal a qualitative transformation of the vector structure at $i \approx 16$. Small $\langle c_{\text{loc}} \rangle$ to the left means that the localization nodes of the first vectors are not synchronized with the large clusters. Small corresponding $\langle \tau_{\text{loc}} \rangle$ shows, however, that the variables x_n and y_n at the localization nodes do not deviate much from their neighborhood. These observations are refined in Fig. 2(c): since $C_1 \approx 1$ and $K_1 \approx 1$ at $i \lesssim 16$, the localization preferably occurs on separated nodes with one connection. Notice finally, see Fig. 2(a), that the vectors with $i \lesssim 16$ correspond to positive λ , and their number is less than the number of positive exponents.

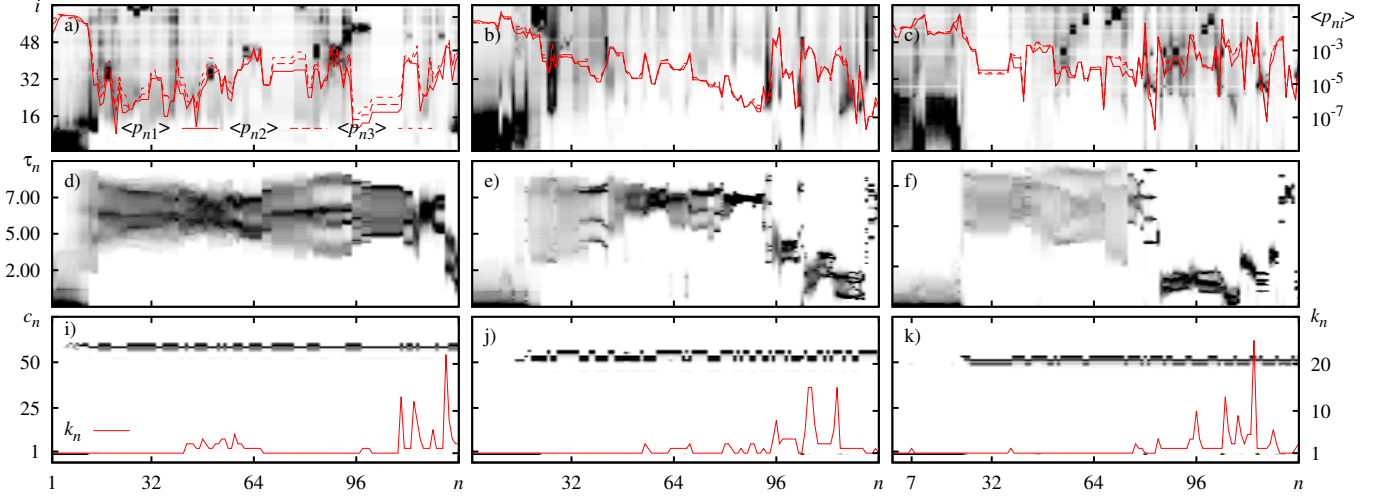


Figure 3. (color online) (a,b,c) $\langle p_{ni} \rangle$ for $\epsilon = 0.13, 0.17$, and 0.22 , respectively. Darker areas correspond to higher values. Red curves represent the first three $\langle p_{ni} \rangle$ in logarithmic scale. (d,e,f) and (i,j,k) Distributions of deviations τ_n and cluster sizes c_n , respectively, corresponding to the above panels. Red curves in panels (i,j,k) represent degree k_n . n and i are the node and the vector indexes, respectively. The nodes are re-numbered as explained in the text. The total computation time is 10^5 , where the cluster detection is performed 10^2 times for trajectory cuts of $\mathcal{T} = 10^3$. $N = 128$.

These observations are typical for the area (I) shown in Fig. 1: the first CLVs always demonstrate high nonwandering localization, such that the nodes $n_{\text{loc}}^{(i)}$ though can change from step to step, always belong to a permanent subset. These CLVs will be referred to as NWL vectors, and the corresponding nodes as NWL nodes. The number of NWL vectors is always less than the number of positive Lyapunov exponents. The nonwandering localization occurs due to the inhomogeneous structure of the considered system. The nonwandering localization of the first CLV was already reported for a disordered medium in Ref. [9].

If a localization site of a vectors freely drifts all over the system it obviously results in a homogeneous picture when the vector is averaged in time. Figure 3(a,b,c) shows that contrary to this the first averaged vectors $\langle p_{ni} \rangle$ have only a few essentially nonzero sites that allows to identify them as NWL vectors. The corresponding NWL nodes are gathered together because of the re-numeration explained below. Red curves represent the first three averaged vectors. One can see that the values at the localization nodes are of two or three orders higher than in the others. Notice also almost identical structures of the NWL vectors.

Gray scales in Fig. 3(d,e,f) represent the distributions of deviations τ_n computed simultaneously with the CLVs above. Observe that the distributions for NWL and non-NWL nodes differ qualitatively. The NWL distributions monotonically decay from the maxima at the origin, while other distributions are separated from zero and have more complicated structure.

The specific form of the distributions for τ_n is employed in Fig. 3 to re-numerate the network and gather the NWL

nodes. First we find a node with the highest maximum of the distribution at the origin. Then treating the distributions as vectors we find a node whose distribution has the smallest euclidean distance to the first one. After that we find the next node as the closest to the second one and so on.

Figure 3(i,j,k) shows the size distributions of clusters with which the nodes are synchronized. These plots are computed simultaneously with the above panels. Observe the correspondence: the NWL nodes preferably do not belong to any cluster, i.e., $c = 1$, while non-NWL ones form large clusters. The red curves demonstrate the degrees of the nodes: the NWL ones typically have only one connection, however in the very rare cases the degree can be 2. One can see only one such node k_7 in the panel (k).

Figure 4(a) represents the distributions of τ_n shown in Fig. 3(e). The nodes at $1 \leq n \leq 14$ (red lines) have almost identical power law distributions. They can be treated as core NWL nodes. Subsequent nodes at $15 \leq n \leq 24$ (blue lines) are intermediate: the distribution exponents at the origin remain unaltered, but humps emerge in the right parts. The node $n = 25$ (green line) is the first non-NWL node, since its distribution vanishes at the origin. Figure 4(b) shows the corresponding distributions of the cluster sizes ρ_c , also demonstrated in Fig. 3(j). One can see that the core NWL nodes are not synchronized with any clusters, while the intermediate ones can sometimes belong to large clusters. It corresponds to the emergence of humps in the panel (a). Finally, the non-NWL nodes always belong to the large clusters.

The NWL vectors can be identified as vectors with $1 \leq i \leq V_{\text{NWL}}$ such that $C_1(i) > 0.5$, where C_1 is the frequency of localization at separated nodes, see the curve

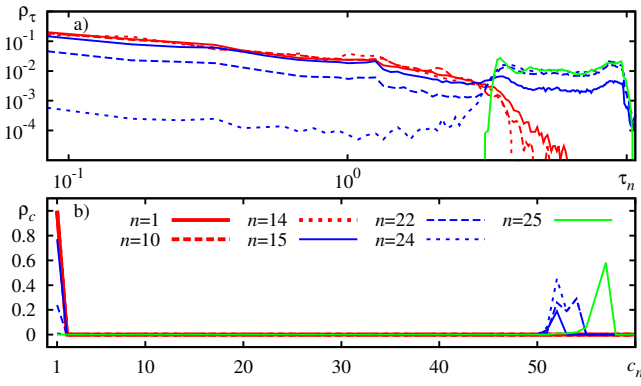


Figure 4. (color online) (a,b) Distributions of deviations τ_n and cluster sizes c_n , respectively, corresponding to Fig. 3(e,j) ($\epsilon = 0.17$). ρ_τ is plotted in double logarithmic scale. Node numbers and corresponding line types are shown in panel (b).

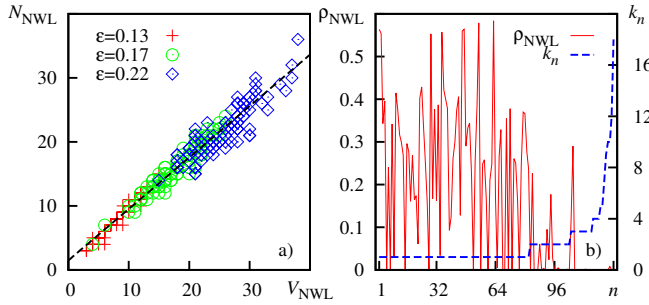


Figure 5. (color online) a) Number of NWL vectors V_{NWL} vs. the number of NWL nodes N_{NWL} . A new matrix \mathbf{A} is generated for each point. Total computation time for each point is 10^5 , where 10^2 synchronization cluster detections are done for $\mathcal{T} = 10^3$. 100 points is computed for each ϵ . The slope of the approximating line is 0.80. b) ρ_{NWL} is the frequency of being not synchronized with any cluster. 1000 runs are performed for the single \mathbf{A} with different initial conditions. $\epsilon = 0.22$, $\langle N_{NWL} \rangle = 22$, $N = 128$.

$C_1(i)$ in Fig. 2(c). The NWL nodes can be found as those which are preferably not synchronized with others, i.e., $\rho_c(1) > 0.5$. Figure 5(a) shows the relation between the number of NWL vectors V_{NWL} and the corresponding number of NWL nodes N_{NWL} . The points in the plot are computed for three values of ϵ and a new \mathbf{A} is generated for each point. Though different ϵ and matrices produces different numbers V_{NWL} and N_{NWL} , the overall data can be approximated well by a straight line, and the slope is 0.8.

Figure 5(b) shows the relative frequency for nodes to belong to the NWL pool. The plot is computed for a single matrix \mathbf{A} using different initial conditions. The nodes are re-numbered according to the ascending order of k_n . On average, the number of NWL nodes for this particular matrix is $\langle N_{NWL} \rangle = 22$, but the distribution occupies a much wider area, including all low connected nodes. It means

that the system selects the NWL nodes at each run according to initial conditions. However only low connected nodes can take part.

Concluding remarks. — We showed that the first CLVs for a chaotic dynamical network with a scale-free structure can demonstrate high nonwandering localization, and the localization nodes can be found without the computation of CLVs. These vectors which we refer to as NWL vectors being averaged in time have almost identical structure and their number always less than the number of positive Lyapunov exponents. Their localization nodes, the NWL nodes, can be identified due to the following features: they are not synchronized with others; they have very low degree (only one connection, in the most cases); they demonstrate identical power law distributions of square deviations of dynamical variables from their neighborhood.

Since by the definition CLV localization sites indicate the most unstable areas of a system, the revealed predictable localization provides an effective criterion for the detection of the most unstable network nodes without an “expensive” computation of the vectors. This is an example of explicit relations between dynamics of a system and the associated tangent space dynamics.

The authors thank U. Parlitz for stimulating discussions.

PK acknowledges the President RF program of support of leading Russian research schools NSh-1726.2014.2.

- [1] K. Kaneko, *Physica D* **23**, 436 (1986).
- [2] G. Giacomelli and A. Politi, *Europhys. Lett.* **15**, 387 (1991).
- [3] M. Falcioni, U. M. B. Marconi, and A. Vulpiani, *Phys. Rev. A* **44**, 2263 (1991).
- [4] F. Ginelli, P. Poggi, A. Turchi, H. Chaté, R. Livi, and A. Politi, *Phys. Rev. Lett.* **99**, 130601 (2007).
- [5] C. L. Wolfe and R. M. Samelson, *Tellus A* **59**, 355–366 (2007).
- [6] R. Buizza and T. N. Palmer, *J. Atmos. Sci.* **52**, 1434 (1995).
- [7] A. S. Pikovsky and J. Kurths, *Phys. Rev. E* **49**, 898 (1994); A. S. Pikovsky and A. Politi, *Nonlinearity* **11**, 1049 (1998).
- [8] D. Pazó, I. G. Szendro, J. M. López, and M. A. Rodríguez, *Phys. Rev. E* **78**, 016209 (2008).
- [9] I. G. Szendro, J. M. López, and M. A. Rodríguez, *Phys. Rev. E* **78**, 036202 (2008).
- [10] A. Politi and A. Torcini, *Chaos* **2**, 293 (1992).
- [11] A.-L. Barabási, R. Albert, and H. Jeong, *Physica A* **281**, 69 (2000).
- [12] S. Jalan, R. E. Amritkar, and C.-K. Hu, *Phys. Rev. E* **72**, 016211 (2005).
- [13] P. V. Kuptsov and U. Parlitz, *J. Nonlinear Sci.* **22**, 727 (2012).
- [14] K. A. Takeuchi, F. Ginelli, and H. Chaté, *Phys. Rev. Lett.* **103**, 154103 (2009).
- [15] G. P. Morriss, *Phys. Rev. E* **85**, 056219 (2012).
- [16] A. Rényi, in *Proc. Fourth Berkeley Symp. on Math. Statist. and Prob.*, Vol. 1 (Univ. of Calif. Press, 1961) pp. 547–561.